Standardised Competence-Oriented Written
School-Leaving Examination/
School-Leaving and Diploma Examination

Formula Booklet

Mathematics (AHS)
Applied Mathematics (BHS)
Higher Education Entrance Examination

Bundesministerium
Bildung, Wissenschaft
und Forschung

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Table of Contents

1 Sets

$\in$	is an element of...
$\notin$	is not an element of...
$\cap$	intersection
$\cup$	union
$\subset$	proper subset
$\subseteq$	subset
$\setminus$	difference ("without")
$\{\ \}$	empty set

Sets of numbers

$\mathbb{N} = \{0, 1, 2, ...\}$	natural numbers
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbb{R}^+$ or $\mathbb{R}^-$	positive or negative real numbers
$\mathbb{R}_0^+$ or $\mathbb{R}_0^-$	positive or negative real numbers including zero

2 Prefixes

tera-	T	10^{12}	deci-	d	10^{-1}
giga-	G	10^{9}	centi-	c	10^{-2}
mega-	M	10^{6}	milli-	m	10^{-3}
kilo-	k	10^{3}	micro-	µ	10^{-6}
hecto-	h	10^{2}	nano-	n	10^{-9}
deca-	da	10^{1}	pico-	p	10^{-12}

3 Powers

Powers with integer exponents

$a \in \mathbb{R};\ n \in \mathbb{N}\setminus\{0\}$

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}} \qquad a^1 = a$$

$a \in \mathbb{R}\setminus\{0\};\ n \in \mathbb{N}\setminus\{0\}$

$$a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n \qquad a^{-1} = \frac{1}{a} \qquad a^0 = 1$$

Powers with rational exponents (roots)

$a, b \in \mathbb{R}_0^+;\ n, k \in \mathbb{N}\setminus\{0\}$ where $n \geq 2$

$$a = \sqrt[n]{b} \quad \Leftrightarrow \quad a^n = b \qquad a^{\frac{1}{n}} = \sqrt[n]{a} \qquad a^{\frac{k}{n}} = \sqrt[n]{a^k} \qquad a^{-\frac{k}{n}} = \frac{1}{\sqrt[n]{a^k}} \quad \text{where} \quad a > 0$$

Calculation rules

$a, b \in \mathbb{R}\setminus\{0\}; r, s \in \mathbb{Z}$ or $a, b \in \mathbb{R}^+; r, s \in \mathbb{Q}$	$a, b \in \mathbb{R}_0^+; m, n, k \in \mathbb{N}\setminus\{0\}$ where $m, n \geq 2$

$a^r \cdot a^s = a^{r+s}$

$\frac{a^r}{a^s} = a^{r-s}$

$(a^r)^s = a^{r \cdot s}$

$(a \cdot b)^r = a^r \cdot b^r$

$\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$

$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

$\sqrt[n]{a^k} = \left(\sqrt[n]{a}\right)^k$

$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ $(b \neq 0)$

$\sqrt[n]{\sqrt[m]{a}} = \sqrt[n \cdot m]{a}$

Binomial formulae

$a, b \in \mathbb{R}; n \in \mathbb{N}$

$(a + b)^2 = a^2 + 2 \cdot a \cdot b + b^2$

$(a - b)^2 = a^2 - 2 \cdot a \cdot b + b^2$

$(a + b) \cdot (a - b) = a^2 - b^2$

$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot a^{n-k} \cdot b^k$

$(a - b)^n = \sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} \cdot a^{n-k} \cdot b^k$

4 Logarithms

$a, b, c \in \mathbb{R}^+$ where $a \neq 1; x, r \in \mathbb{R}$

$x = \log_a(b) \quad \Leftrightarrow \quad a^x = b$

$\log_a(b \cdot c) = \log_a(b) + \log_a(c)$ $\qquad$ $\log_a\left(\frac{b}{c}\right) = \log_a(b) - \log_a(c)$ $\qquad$ $\log_a(b^r) = r \cdot \log_a(b)$

$\log_a(a^x) = x$ $\qquad$ $\log_a(a) = 1$ $\qquad$ $\log_a(1) = 0$ $\qquad$ $\log_a\left(\frac{1}{a}\right) = -1$ $\qquad$ $a^{\log_a(b)} = b$

natural logarithm (logarithm with base e): $\ln(b) = \log_e(b)$
common logarithm (logarithm with base 10): $\lg(b) = \log_{10}(b)$

5 Quadratic Equations

$p, q \in \mathbb{R}$	$a, b, c \in \mathbb{R}$ where $a \neq 0$

$x^2 + p \cdot x + q = 0$

$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$

$a \cdot x^2 + b \cdot x + c = 0$

$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$

Vieta's Theorem
x_1 and x_2 are the solutions to the equation $x^2 + p \cdot x + q = 0$ if and only if:
$x_1 + x_2 = -p$
$x_1 \cdot x_2 = q$

Linear factorisation
$x^2 + p \cdot x + q = (x - x_1) \cdot (x - x_2)$

6 Two-Dimensional Shapes

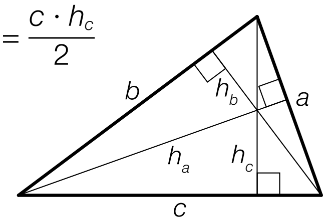
A ... area
u ... perimeter

Triangle

General triangle

$$A = \frac{a \cdot h_a}{2} = \frac{b \cdot h_b}{2} = \frac{c \cdot h_c}{2}$$

$$u = a + b + c$$



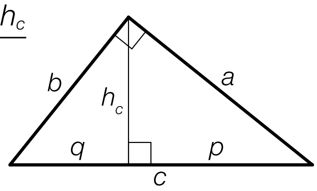
Right-angled triangle with hypotenuse *c* and sides *a*, *b*

$$A = \frac{a \cdot b}{2} = \frac{c \cdot h_c}{2}$$

$$h_c^2 = p \cdot q$$

$$a^2 = c \cdot p$$

$$b^2 = c \cdot q$$



Heron's Formula

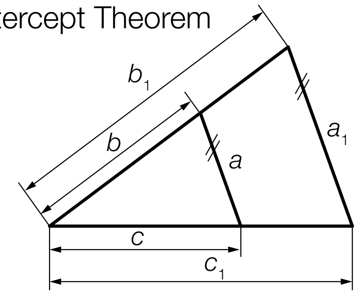
$$A = \sqrt{s \cdot (s - a) \cdot (s - b) \cdot (s - c)} \text{ where } s = \frac{a + b + c}{2}$$

Pythagorean theorem

$$a^2 + b^2 = c^2$$

Similarity and the Intercept Theorem

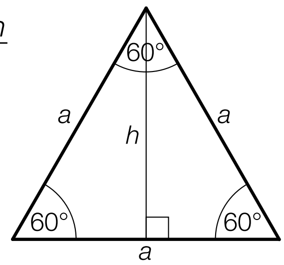
$$\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$$



Equilateral triangle

$$A = \frac{a^2}{4} \cdot \sqrt{3} = \frac{a \cdot h}{2}$$

$$h = \frac{a}{2} \cdot \sqrt{3}$$

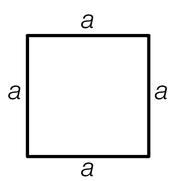


Quadrilateral

Square

$$A = a^2$$

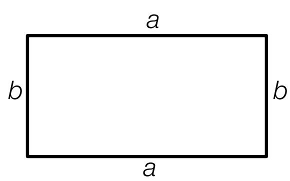
$$u = 4 \cdot a$$



Rectangle

$$A = a \cdot b$$

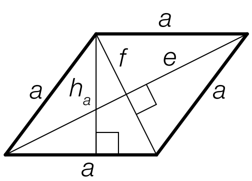
$$u = 2 \cdot a + 2 \cdot b$$



Rhombus

$$A = a \cdot h_a = \frac{e \cdot f}{2}$$

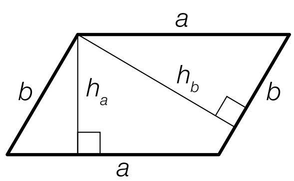
$$u = 4 \cdot a$$



Parallelogram

$$A = a \cdot h_a = b \cdot h_b$$

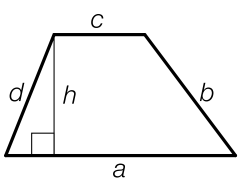
$$u = 2 \cdot a + 2 \cdot b$$



Trapezium

$$A = \frac{(a + c) \cdot h}{2}$$

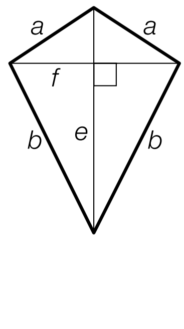
$$u = a + b + c + d$$



Kite

$$A = \frac{e \cdot f}{2}$$

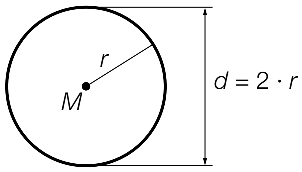
$$u = 2 \cdot a + 2 \cdot b$$



Circle

$A = \pi \cdot r^2 = \dfrac{\pi \cdot d^2}{4}$

$u = 2 \cdot \pi \cdot r = \pi \cdot d$

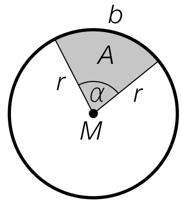


Arc length and sector of a circle

α in degrees (°)

$b = \pi \cdot r \cdot \dfrac{\alpha}{180°}$

$A = \pi \cdot r^2 \cdot \dfrac{\alpha}{360°} = \dfrac{b \cdot r}{2}$



7 Solids

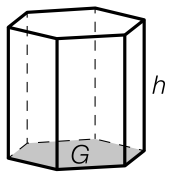
V ... volume	M ... lateral surface area
O ... surface area	u_G ... perimeter of the base
G ... area of the base	

Prism

$V = G \cdot h$

$M = u_G \cdot h$

$O = 2 \cdot G + M$

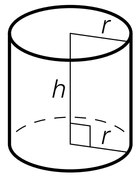


Cylinder

$V = \pi \cdot r^2 \cdot h$

$M = 2 \cdot \pi \cdot r \cdot h$

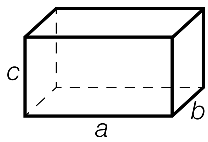
$O = 2 \cdot \pi \cdot r^2 + 2 \cdot \pi \cdot r \cdot h$



Cuboid

$V = a \cdot b \cdot c$

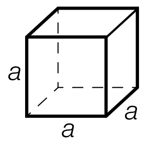
$O = 2 \cdot (a \cdot b + a \cdot c + b \cdot c)$



Cube

$V = a^3$

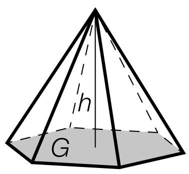
$O = 6 \cdot a^2$



Pyramid

$V = \dfrac{G \cdot h}{3}$

$O = G + M$



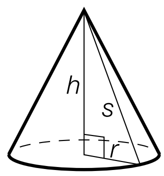
Cone

$V = \dfrac{1}{3} \cdot \pi \cdot r^2 \cdot h$

$M = \pi \cdot r \cdot s$

$O = \pi \cdot r^2 + \pi \cdot r \cdot s$

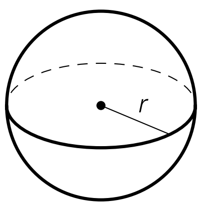
$s = \sqrt{h^2 + r^2}$



Sphere

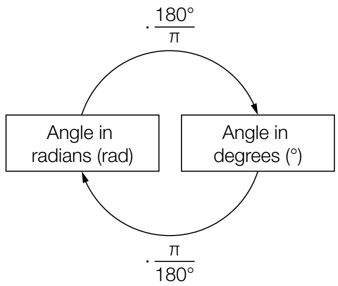
$V = \dfrac{4}{3} \cdot \pi \cdot r^3$

$O = 4 \cdot \pi \cdot r^2$



8 Trigonometry

Converting between degrees and radians

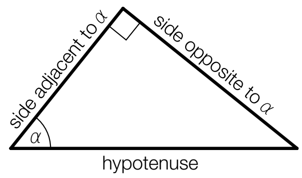


Right-angled triangle trigonometry

Sine: $\sin(\alpha) = \frac{\text{side opposite to } \alpha}{\text{hypotenuse}}$

Cosine: $\cos(\alpha) = \frac{\text{side adjacent to } \alpha}{\text{hypotenuse}}$

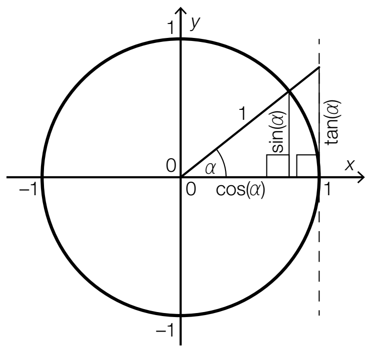
Tangent: $\tan(\alpha) = \frac{\text{side opposite to } \alpha}{\text{side adjacent to } \alpha}$



Unit circle trigonometry

$\sin^2(\alpha) + \cos^2(\alpha) = 1$

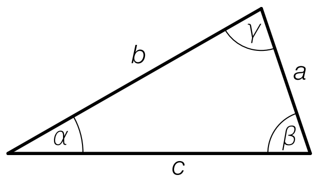
$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$ for $\cos(\alpha) \neq 0$



Trigonometry in general triangles

Sine Rule: $\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$

Cosine Rule: $a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos(\alpha)$
$b^2 = a^2 + c^2 - 2 \cdot a \cdot c \cdot \cos(\beta)$
$c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos(\gamma)$



Trigonometric formula for the area of a triangle

$A = \frac{1}{2} \cdot b \cdot c \cdot \sin(\alpha) = \frac{1}{2} \cdot a \cdot c \cdot \sin(\beta) = \frac{1}{2} \cdot a \cdot b \cdot \sin(\gamma)$

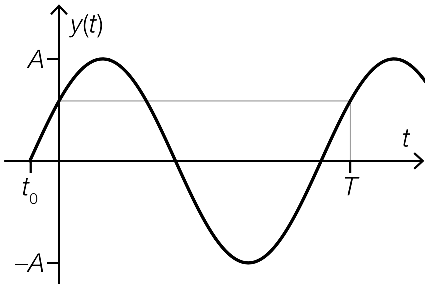
General sine function (in terms of time t)

A ... amplitude
ω ... angular frequency
φ ... zero phase angle
T ... oscillation period (period length)
f ... frequency

$y(t) = A \cdot \sin(\omega \cdot t + \varphi)$

$T = \frac{2 \cdot \pi}{\omega} = \frac{1}{f}$

$t_0 = -\frac{\varphi}{\omega}$



9 Complex Numbers

j or i ... imaginary unit with $j^2 = -1$ or $i^2 = -1$

a ... real part, $a \in \mathbb{R}$

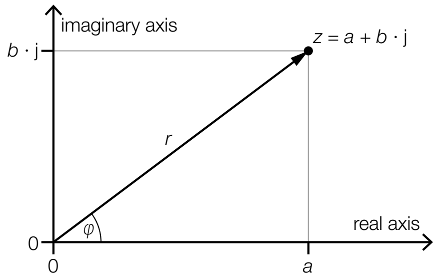
b ... imaginary part, $b \in \mathbb{R}$

r ... modulus, $r \in \mathbb{R}_0^+$

φ ... argument, $\varphi \in \mathbb{R}$

Cartesian form

$z = a + b \cdot j$



Polar forms

$z = r \cdot [\cos(\varphi) + j \cdot \sin(\varphi)] = r \cdot e^{j \cdot \varphi} = (r;\, \varphi) = r \angle \varphi$

Conversions

$a = r \cdot \cos(\varphi)$ $\qquad r = \sqrt{a^2 + b^2}$

$b = r \cdot \sin(\varphi)$ $\qquad \tan(\varphi) = \frac{b}{a}$

10 Vectors

P, Q ... points

Vectors in $\mathbb{R}^2$

Arrow from P to Q:

$P = (p_1, p_2)$, $Q = (q_1, q_2)$

$$\overrightarrow{PQ} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \end{pmatrix}$$

Calculation rules in $\mathbb{R}^2$

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \vec{a} \pm \vec{b} = \begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \end{pmatrix}$$

$$k \cdot \vec{a} = k \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} k \cdot a_1 \\ k \cdot a_2 \end{pmatrix} \quad \text{where} \quad k \in \mathbb{R}$$

Scalar product in $\mathbb{R}^2$

$\vec{a} \cdot \vec{b} = a_1 \cdot b_1 + a_2 \cdot b_2$

Absolute value (length) of a vector in $\mathbb{R}^2$

$|\vec{a}| = \sqrt{{a_1}^2 + {a_2}^2}$

$\vec{n}$, a vector perpendicular to $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ in $\mathbb{R}^2$

$$\vec{n} = k \cdot \begin{pmatrix} -a_2 \\ a_1 \end{pmatrix} \quad \text{for} \quad |\vec{a}| \neq 0 \text{ and } k \in \mathbb{R} \setminus \{0\}$$

Vectors in $\mathbb{R}^n$

Arrow from P to Q:

$P = (p_1, p_2, \ldots, p_n)$, $Q = (q_1, q_2, \ldots, q_n)$

$$\overrightarrow{PQ} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \\ \vdots \\ q_n - p_n \end{pmatrix}$$

Calculation rules in $\mathbb{R}^n$

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \vec{a} \pm \vec{b} = \begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ \vdots \\ a_n \pm b_n \end{pmatrix}$$

$$k \cdot \vec{a} = k \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} k \cdot a_1 \\ k \cdot a_2 \\ \vdots \\ k \cdot a_n \end{pmatrix} \quad \text{where} \quad k \in \mathbb{R}$$

Scalar product in $\mathbb{R}^n$

$\vec{a} \cdot \vec{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + \ldots + a_n \cdot b_n$

Absolute value (length) of a vector in $\mathbb{R}^n$

$|\vec{a}| = \sqrt{{a_1}^2 + {a_2}^2 + \ldots + {a_n}^2}$

Criterion for two vectors to be perpendicular in $\mathbb{R}^2$ and $\mathbb{R}^3$

$\vec{a} \cdot \vec{b} = 0 \quad \Leftrightarrow \quad \vec{a} \perp \vec{b}$ for $|\vec{a}| \neq 0$ and $|\vec{b}| \neq 0$

Criterion for two vectors to be parallel in $\mathbb{R}^2$ and $\mathbb{R}^3$

$\vec{a} \parallel \vec{b} \quad \Leftrightarrow \quad \vec{a} = k \cdot \vec{b}$ for $|\vec{a}| \neq 0, |\vec{b}| \neq 0$ and $k \in \mathbb{R}\backslash\{0\}$

Angle φ between $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^2$ and $\mathbb{R}^3$

$\cos(\varphi) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$ for $|\vec{a}| \neq 0$ and $|\vec{b}| \neq 0$

Unit vector $\vec{a}_0$ in the direction of $\vec{a}$

$\vec{a}_0 = \frac{1}{|\vec{a}|} \cdot \vec{a}$ for $|\vec{a}| \neq 0$

Vector product in $\mathbb{R}^3$

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 \cdot b_3 - a_3 \cdot b_2 \\ a_3 \cdot b_1 - a_1 \cdot b_3 \\ a_1 \cdot b_2 - a_2 \cdot b_1 \end{pmatrix}$$

11 Straight Lines

g ... line	$\vec{g}$ ... a direction vector for the line g $\vec{n}$ ... a vector perpendicular to the line g X, P ... points on the line g m ... gradient of the line g α ... angle of slope of the line g $a, b, c, d, m \in \mathbb{R}$

Vector equation of a line g in $\mathbb{R}^2$ and $\mathbb{R}^3$

$g: X = P + t \cdot \vec{g}$ where $t \in \mathbb{R}$

Equation of a line g in $\mathbb{R}^2$

the explicit equation of a line: $g:\ y = m \cdot x + c$ where $m = \tan(\alpha)$

a general equation of a line: $g:\ a \cdot x + b \cdot y = d$

a normal vector representation: $g:\ \vec{n} \cdot X = \vec{n} \cdot P$

(for both:) where $\vec{n} \parallel \begin{pmatrix} a \\ b \end{pmatrix}$ for $\begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

12 Matrices

$a_{ij}, b_{ij} \in \mathbb{R}; i, j, m, n, p \in \mathbb{N}\backslash\{0\}; k \in \mathbb{R}$

Addition/subtraction of matrices

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \pm \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} \pm b_{11} & \cdots & a_{1n} \pm b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & \cdots & a_{mn} \pm b_{mn} \end{pmatrix}$$

Multiplication of a matrix by a number *k*

$$k \cdot \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} k \cdot a_{11} & \cdots & k \cdot a_{1n} \\ \vdots & \ddots & \vdots \\ k \cdot a_{m1} & \cdots & k \cdot a_{mn} \end{pmatrix}$$

Matrix multiplication

$\boldsymbol{A}$ ... $m \times p$-matrix $\quad$ $\boldsymbol{B}$ ... $p \times n$-matrix $\quad$ $\boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B}$ ... $m \times n$-matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ip} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mp} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ \vdots & & \vdots & & \vdots \\ b_{p1} & \cdots & b_{pj} & \cdots & b_{pn} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{in} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mn} \end{pmatrix}$$

where $c_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \ldots + a_{ip} \cdot b_{pj}$

Identity matrix *I*

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Transposed matrix $\boldsymbol{A}^{\mathrm{T}}$

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$\boldsymbol{A}^{\mathrm{T}} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Inverse matrix $\boldsymbol{A}^{-1}$ of a square matrix *A*

$$\boldsymbol{A} \cdot \boldsymbol{A}^{-1} = \boldsymbol{A}^{-1} \cdot \boldsymbol{A} = \boldsymbol{I}$$

Systems of linear equations in matrix notation (*n* equations with *n* unknowns)

$a_{11} \cdot x_1 + a_{12} \cdot x_2 + \ldots + a_{1n} \cdot x_n = b_1$
$a_{21} \cdot x_1 + a_{22} \cdot x_2 + \ldots + a_{2n} \cdot x_n = b_2$
$\ldots$
$a_{n1} \cdot x_1 + a_{n2} \cdot x_2 + \ldots + a_{nn} \cdot x_n = b_n$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{\vec{b}}$$

If the inverse matrix $\boldsymbol{A}^{-1}$ exists, then $\vec{x} = \boldsymbol{A}^{-1} \cdot \vec{b}$ holds

Manufacturing processes

$\boldsymbol{A}$ ... square material consumption matrix
$\boldsymbol{I}$ ... identity matrix
$\vec{\boldsymbol{x}}$ ... production vector
$\vec{\boldsymbol{n}}$ ... demand vector

$\vec{\boldsymbol{x}} = \boldsymbol{A} \cdot \vec{\boldsymbol{x}} + \vec{\boldsymbol{n}}$ $\qquad$ $\vec{\boldsymbol{x}} = (\boldsymbol{I} - \boldsymbol{A})^{-1} \cdot \vec{\boldsymbol{n}}$ $\qquad$ $\vec{\boldsymbol{n}} = (\boldsymbol{I} - \boldsymbol{A}) \cdot \vec{\boldsymbol{x}}$

13 Sequences and Series

Arithmetic sequence

$(a_n) = (a_1, a_2, a_3, ...)$

$d = a_{n+1} - a_n$

Recursive rule

$a_{n+1} = a_n + d$ with a_1 given

Explicit rule

$a_n = a_1 + (n-1) \cdot d$

Geometric sequence

$(b_n) = (b_1, b_2, b_3, ...)$

$q = \frac{b_{n+1}}{b_n}$

Recursive rule

$b_{n+1} = b_n \cdot q$ with b_1 given

Explicit rule

$b_n = b_1 \cdot q^{n-1}$

Finite arithmetic series

Sum s_n of the first n terms

$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_{n-1} + a_n$

$s_n = \frac{n}{2} \cdot (a_1 + a_n) = \frac{n}{2} \cdot [2 \cdot a_1 + (n-1) \cdot d]$

Finite geometric series

Sum s_n of the first n terms

$s_n = \sum_{i=1}^{n} b_i = b_1 + b_2 + ... + b_{n-1} + b_n$

$s_n = b_1 \cdot \frac{q^n - 1}{q - 1}$ for $q \neq 1$

Infinite geometric series

$\sum_{n=1}^{\infty} b_n$ is convergent if and only if $|q| < 1$

$s = \lim_{n \to \infty} s_n = \frac{b_1}{1-q}$ for $|q| < 1$

14 Rates of Change

For a real function f defined over an interval $[a, b]$:

Absolute change of f in $[a, b]$

$f(b) - f(a)$

Relative (percentage) change of f in $[a, b]$

$\frac{f(b) - f(a)}{f(a)}$ for $f(a) \neq 0$

Difference quotient (average rate of change) of f in $[a, b]$ or in $[x, x + \Delta x]$

$\frac{f(b) - f(a)}{b - a}$ or $\frac{f(x + \Delta x) - f(x)}{\Delta x}$ for $b \neq a$ or $\Delta x \neq 0$

Differential quotient (instantaneous rate of change) of f at the point x

$f'(x) = \lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x}$ or $f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

15 Growth and Decay Processes

t ... time
$N(t)$ ... amount at time t
$N_0 = N(0)$ ... amount at time $t = 0$

Linear

$k \in \mathbb{R}^+$

linear growth	$N(t) = N_0 + k \cdot t$	
linear decay	$N(t) = N_0 - k \cdot t$	

Exponential

$a, \lambda \in \mathbb{R}^+$ where $a \neq 1$ and $N_0 > 0$
a ... growth factor

exponential growth	$N(t) = N_0 \cdot a^t$ for $a > 1$	$N(t) = N_0 \cdot e^{\lambda \cdot t}$
exponential decay	$N(t) = N_0 \cdot a^t$ for $0 < a < 1$	$N(t) = N_0 \cdot e^{-\lambda \cdot t}$

Limited

$S, a, \lambda \in \mathbb{R}^+$ where $0 < a < 1$
S ... saturation value, carrying capacity

limited growth (saturation function)	$N(t) = S - b \cdot a^t$ where $b = S - N_0$	$N(t) = S - b \cdot e^{-\lambda \cdot t}$ where $b = S - N_0$
limited decay	$N(t) = S + b \cdot a^t$ where $b = \lvert S - N_0 \rvert$	$N(t) = S + b \cdot e^{-\lambda \cdot t}$ where $b = \lvert S - N_0 \rvert$

Logistic

$S, a, \lambda \in \mathbb{R}^+$ where $0 < a < 1$ and $N_0 > 0$
S ... saturation value, carrying capacity

logistic growth	$N(t) = \dfrac{S}{1 + c \cdot a^t}$ where $c = \dfrac{S - N_0}{N_0}$	$N(t) = \dfrac{S}{1 + c \cdot e^{-\lambda \cdot t}}$ where $c = \dfrac{S - N_0}{N_0}$

16 Differentiation and Integration

f, g, h ... functions that are differentiable over $\mathbb{R}$ or over a defined interval
f', g', h' ... derivative functions
F, G, H ... antiderivatives
$C, k, q \in \mathbb{R}$; $a \in \mathbb{R}^+\backslash\{1\}$

Indefinite integral

$\int f(x)\,\mathrm{d}x = F(x) + C \quad$ where $\quad F' = f$

Definite integral

$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a)$

Function	Derivative	Antiderivative
$f(x) = k$	$f'(x) = 0$	$F(x) = k \cdot x$
$f(x) = x^q$	$f'(x) = q \cdot x^{q-1}$	$F(x) = \frac{x^{q+1}}{q+1}$ for $q \neq -1$ $F(x) = \ln(\lvert x \rvert)$ for $q = -1$
$f(x) = e^x$	$f'(x) = e^x$	$F(x) = e^x$
$f(x) = a^x$	$f'(x) = \ln(a) \cdot a^x$	$F(x) = \frac{a^x}{\ln(a)}$
$f(x) = \ln(x)$	$f'(x) = \frac{1}{x}$	$F(x) = x \cdot \ln(x) - x$
$f(x) = \log_a(x)$	$f'(x) = \frac{1}{x \cdot \ln(a)}$	$F(x) = \frac{1}{\ln(a)} \cdot (x \cdot \ln(x) - x)$
$f(x) = \sin(x)$	$f'(x) = \cos(x)$	$F(x) = -\cos(x)$
$f(x) = \cos(x)$	$f'(x) = -\sin(x)$	$F(x) = \sin(x)$
$f(x) = \tan(x)$	$f'(x) = 1 + \tan^2(x) = \frac{1}{\cos^2(x)}$	$F(x) = -\ln(\lvert \cos(x) \rvert)$
$g(x) = k \cdot f(x)$	$g'(x) = k \cdot f'(x)$	$G(x) = k \cdot F(x)$
$h(x) = f(x) \pm g(x)$	$h'(x) = f'(x) \pm g'(x)$	$H(x) = F(x) \pm G(x)$
$g(x) = f(k \cdot x)$	$g'(x) = k \cdot f'(k \cdot x)$	$G(x) = \frac{1}{k} \cdot F(k \cdot x)$

Differentiation rules

multiplication by a constant	$(k \cdot f)' = k \cdot f'$
sum rule	$(f \pm g)' = f' \pm g'$
product rule	$(f \cdot g)' = f' \cdot g + f \cdot g'$
quotient rule	$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$ for $g(x) \neq 0$
chain rule	$h(x) = f(g(x)) \quad \Rightarrow \quad h'(x) = f'(g(x)) \cdot g'(x)$

Method for integration – linear substitution

$$\int f(a \cdot x + b)\,\mathrm{d}x = \frac{F(a \cdot x + b)}{a} + C$$

Volume V of solids of revolution

Rotation of the graph of a function f with $y = f(x)$ about an axis

Rotation about the x-axis ($a \leq x \leq b$)

$$V_x = \pi \cdot \int_a^b y^2 \,\mathrm{d}x$$

Rotation about the y-axis ($c \leq y \leq d$)

$$V_y = \pi \cdot \int_c^d x^2 \,\mathrm{d}y$$

Arc length s of the graph of a function f in the interval $[a, b]$

$$s = \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x$$

Mean m of a function f in the interval $[a, b]$

$$m = \frac{1}{b - a} \cdot \int_a^b f(x)\,\mathrm{d}x$$

17 1st Order Differential Equations

Separable differential equations

$y' = f(x) \cdot g(y)$ or $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x) \cdot g(y)$ where $y = y(x)$

1st order linear differential equation with constant coefficients

y ... general solution of a nonhomogeneous differential equation
y_h ... general solution of the homogeneous differential equation $y' + a \cdot y = 0$
y_p ... particular solution of the nonhomogeneous differential equation
s ... interference function

$y' + a \cdot y = s(x)$ where $a \in \mathbb{R}$, $y = y(x)$

$y = y_\mathrm{h} + y_\mathrm{p}$

18 Statistics

$x_1, x_2, \dots, x_n$ ... a list of n real numbers
for which k different values $x_1, x_2, \dots, x_k$ occur.
H_i ... absolute frequency of x_i with $H_1 + H_2 + \dots + H_k = n$

Relative frequency h_i of x_i

$$h_i = \frac{H_i}{n}$$

Measures of central tendency

Arithmetic mean $\bar{x}$

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$$

$$\bar{x} = \frac{x_1 \cdot H_1 + x_2 \cdot H_2 + \dots + x_k \cdot H_k}{n} = \frac{1}{n} \cdot \sum_{i=1}^{k} x_i \cdot H_i$$

Geometric mean $\bar{x}_{\text{geo}}$

$$\bar{x}_{\text{geo}} = \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} \quad \text{for} \quad x_i > 0$$

Median $\tilde{x}$ for metric data

$x_{(1)} \le x_{(2)} \le \dots \le x_{(n)}$ ... ordered list of n values

$$\tilde{x} = \begin{cases} x_{\left(\frac{n+1}{2}\right)} & \text{... when } n \text{ is odd} \\ \frac{1}{2} \cdot \left(x_{\left(\frac{n}{2}\right)} + x_{\left(\frac{n}{2}+1\right)}\right) & \text{... when } n \text{ is even} \end{cases}$$

Quartiles

q_1: At least 25 % of the values are less than or equal to q_1, and at least 75 % of the values are greater than or equal to q_1.

$q_2 = \tilde{x}$: At least 50 % of the values are less than or equal to q_2, and at least 50 % of the values are greater than or equal to q_2.

q_3: At least 75 % of the values are less than or equal to q_3, and at least 25 % of the values are greater than or equal to q_3.

Measures of spread

Range: $x_{\max} - x_{\min}$

Interquartile range: $q_3 - q_1$

s^2 ... (empirical) variance of a sample
s ... (empirical) standard deviation of a sample

$$s^2 = \frac{1}{n} \cdot \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$s = \sqrt{\frac{1}{n} \cdot \sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$s^2 = \frac{1}{n} \cdot \sum_{i=1}^{k} (x_i - \bar{x})^2 \cdot H_i$$

$$s = \sqrt{\frac{1}{n} \cdot \sum_{i=1}^{k} (x_i - \bar{x})^2 \cdot H_i}$$

If the variance of a population should be estimated using a sample of size n.

$$s_{n-1}^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$s_{n-1} = \sqrt{\frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$s_{n-1}^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{k} (x_i - \bar{x})^2 \cdot H_i$$

$$s_{n-1} = \sqrt{\frac{1}{n-1} \cdot \sum_{i=1}^{k} (x_i - \bar{x})^2 \cdot H_i}$$

19 Probability

$n \in \mathbb{N}\backslash\{0\}$; $k \in \mathbb{N}$ where $k \leq n$
A, B ... events
$\overline{A}$ or $\neg A$ ... complementary event of A
$A \cap B$ or $A \wedge B$ ... A and B (the event A and the event B both occur)
$A \cup B$ or $A \vee B$ ... A or B (at least one of the two events A or B occurs)
$P(A)$ ... probability of event A occurring
$P(A|B)$ ... probability of event A occurring given that event B has occurred (conditional probability)

Factorial

$n! = n \cdot (n-1) \cdot ... \cdot 1$ $\quad 0! = 1 \quad 1! = 1$

Binomial coefficient

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

Probability for a Laplace experiment

$$P(A) = \frac{\text{number of successful outcomes for } A}{\text{number of possible outcomes}}$$

Elementary rules

$P(\overline{A}) = 1 - P(A)$ or $P(\neg A) = 1 - P(A)$

$P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$ or $P(A \wedge B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$

If A and B are (stochastically) independent of one another:
$P(A \cap B) = P(A) \cdot P(B)$ or $P(A \wedge B) = P(A) \cdot P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ or $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$

If A and B are mutually exclusive:
$P(A \cup B) = P(A) + P(B)$ or $P(A \vee B) = P(A) + P(B)$

Conditional probability of *A* given the condition *B*

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ or $P(A|B) = \frac{P(A \wedge B)}{P(B)}$

Bayes' Theorem

$$P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)} = \frac{P(A) \cdot P(B|A)}{P(A) \cdot P(B|A) + P(\overline{A}) \cdot P(B|\overline{A})}$$

or

$$P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)} = \frac{P(A) \cdot P(B|A)}{P(A) \cdot P(B|A) + P(\neg A) \cdot P(B|\neg A)}$$

Expectation value μ of a discrete random variable X with values $x_1, x_2, \dots, x_n$

$$\mu = E(X) = x_1 \cdot P(X = x_1) + x_2 \cdot P(X = x_2) + \dots + x_n \cdot P(X = x_n) = \sum_{i=1}^{n} x_i \cdot P(X = x_i)$$

Variance σ^2 of a discrete random variable X with values $x_1, x_2, \dots, x_n$

$$\sigma^2 = V(X) = \sum_{i=1}^{n} (x_i - \mu)^2 \cdot P(X = x_i)$$

Standard deviation σ

$$\sigma = \sqrt{V(X)}$$

Binomial distribution

$n \in \mathbb{N}\backslash\{0\}; k \in \mathbb{N}; p \in \mathbb{R}$ where $k \leq n$ and $0 \leq p \leq 1$

The random variable X is binomially distributed with parameters n and p

$$P(X = k) = \binom{n}{k} \cdot p^k \cdot (1 - p)^{n-k}$$

Expectation value: $E(X) = \mu = n \cdot p$
Variance: $V(X) = \sigma^2 = n \cdot p \cdot (1 - p)$

Normal distribution

$\mu, \sigma \in \mathbb{R}$ where $\sigma > 0$
f ... probability density function
F ... cumulative distribution function
φ ... probability density function of the standard normal distribution
ϕ ... cumulative density function of the standard normal distribution

Normal distribution $N(\mu; \sigma^2)$: The random variable X is normally distributed with expectation value (μ), standard deviation (σ) and variance (σ^2)

$$P(X \leq x_1) = F(x_1) = \int_{-\infty}^{x_1} f(x)\,\mathrm{d}x = \int_{-\infty}^{x_1} \frac{1}{\sigma \cdot \sqrt{2 \cdot \pi}} \cdot e^{-\frac{1}{2} \cdot \left(\frac{x-\mu}{\sigma}\right)^2}\,\mathrm{d}x$$

Probabilities for the empirical rule
$P(\mu - \sigma \leq X \leq \mu + \sigma) \approx 0.683$
$P(\mu - 2 \cdot \sigma \leq X \leq \mu + 2 \cdot \sigma) \approx 0.954$
$P(\mu - 3 \cdot \sigma \leq X \leq \mu + 3 \cdot \sigma) \approx 0.997$

Standard normal distribution $N(0, 1)$

$$z = \frac{x - \mu}{\sigma}$$

$$\phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \varphi(x)\,\mathrm{d}x = \frac{1}{\sqrt{2 \cdot \pi}} \cdot \int_{-\infty}^{z} e^{-\frac{x^2}{2}}\,\mathrm{d}x$$

$$\phi(-z) = 1 - \phi(z)$$

$$P(-z \leq Z \leq z) = 2 \cdot \phi(z) - 1$$

$P(-z \leq Z \leq z)$	= 90 %	= 95 %	= 99 %
z	≈ 1.645	≈ 1.960	≈ 2.576

Prediction Intervals and Confidence Intervals

$\mu, \sigma, \alpha \in \mathbb{R}$ where $\sigma > 0$ and $0 < \alpha < 1$
$\bar{x}$ ... sample mean
s_{n-1} ... sample standard deviation
n ... sample size
$z_{1-\frac{\alpha}{2}}$ ... $\left(1-\frac{\alpha}{2}\right)$-quantile of the standard normal distribution
$t_{f;1-\frac{\alpha}{2}}$ ... $\left(1-\frac{\alpha}{2}\right)$-quantile of the t-distribution with f degrees of freedom

Two-sided $(1-\alpha)$-prediction interval for a single value of a normally distributed random variable

$$\left[\mu - z_{1-\frac{\alpha}{2}} \cdot \sigma,\ \mu + z_{1-\frac{\alpha}{2}} \cdot \sigma\right]$$

Two-sided $(1-\alpha)$-prediction interval for the sample mean of normally distributed values

$$\left[\mu - z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}},\ \mu + z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right]$$

Two-sided $(1-\alpha)$-confidence interval for the expectation value of a normally distributed random variable

known σ: $\left[\bar{x} - z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}},\ \bar{x} + z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right]$

unknown σ: $\left[\bar{x} - t_{f,1-\frac{\alpha}{2}} \cdot \frac{s_{n-1}}{\sqrt{n}},\ \bar{x} + t_{f,1-\frac{\alpha}{2}} \cdot \frac{s_{n-1}}{\sqrt{n}}\right]$ where $f = n - 1$

20 Linear Regression

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ ... pairs of values
$\bar{x}, \bar{y}$ ... mean of x_i and y_i

linear regression function f with $f(x) = m \cdot x + c$

$$m = \frac{\sum_{i=1}^{n} (x_i - \bar{x}) \cdot (y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$c = \bar{y} - m \cdot \bar{x}$$

Pearson's correlation coefficient

$$r = \frac{\sum_{i=1}^{n} (x_i - \bar{x}) \cdot (y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2 \cdot \sum_{i=1}^{n} (y_i - \bar{y})^2}}$$

21 Financial Mathematics

Compound interest calculation

K_0 ... initial investment
K_n ... final capital after n years
i ... annual percentage rate of interest

simple interest: $K_n = K_0 \cdot (1 + i \cdot n)$
compound interest: $K_n = K_0 \cdot (1 + i)^n$

Interest calculated during the year

m ... number of compounding periods per year

The following abbreviations are used for compounding periods:
p. a. ... per year
p. s. ... per semester
p. q. ... per quarter
p. m. ... per month

$$K_n = K_0 \cdot (1 + i_m)^{n \cdot m}$$

interest rate applied during the year i_m ← equivalent interest rates → effective annual interest rate i

$$i_m = \sqrt[m]{1 + i} - 1$$

$$i = (1 + i_m)^m - 1$$

Annuities

R ... amount paid per time period
n ... number of payments
i ... interest rate
$q = 1 + i$ ... accumulation factor

Requirement: annuity period = interest period

	ordinary annuity	annuity due
final value E	$E_{\text{ordinary}} = R \cdot \frac{q^n - 1}{q - 1}$	$E_{\text{due}} = R \cdot \frac{q^n - 1}{q - 1} \cdot q$
present value B	$B_{\text{ordinary}} = R \cdot \frac{q^n - 1}{q - 1} \cdot \frac{1}{q^n}$	$B_{\text{due}} = R \cdot \frac{q^n - 1}{q - 1} \cdot \frac{1}{q^{n-1}}$

Amortisation table

period	interest amount	repayment amount	annuity	residual debt
0				K_0
1	$K_0 \cdot i$	T_1	$A_1 = K_0 \cdot i + T_1$	$K_1 = K_0 - T_1$
...	...	...	...	...

22 Investments

E_t ... revenue in year t
A_t ... expenses in year t
A_0 ... acquisition costs
R_t ... returns in year t
i ... imputed interest rate (annual interest rate)
n ... operating duration in years
i_w ... reinvestment interest rate (annual interest rate)
E ... final value of the reinvested returns

$$R_t = E_t - A_t$$

Net present value C_0

$$C_0 = -A_0 + \left[\frac{R_1}{(1+i)} + \frac{R_2}{(1+i)^2} + \ldots + \frac{R_n}{(1+i)^n}\right]$$

Internal rate of return i_{internal}

$$-A_0 + \left[\frac{R_1}{(1+i_{\text{internal}})} + \frac{R_2}{(1+i_{\text{internal}})^2} + \ldots + \frac{R_n}{(1+i_{\text{internal}})^n}\right] = 0$$

Modified internal rate of return i_{mod}

$$A_0 \cdot (1+i_{\text{mod}})^n = E \quad \text{where} \quad E = R_1 \cdot (1+i_w)^{n-1} + R_2 \cdot (1+i_w)^{n-2} + \ldots + R_{n-1} \cdot (1+i_w) + R_n$$

23 Cost-of-Production and Theory of Value

x ... amount produced, offered, required or sold ($x \geq 0$)

cost function K	$K(x)$
fixed costs F	$K(0)$
variable cost function K_v	$K_v(x) = K(x) - F$
marginal cost function K'	$K'(x)$
unit cost function (average cost function) $\overline{K}$	$\overline{K}(x) = \frac{K(x)}{x}$
variable unit cost function (variable average cost function) $\overline{K_v}$	$\overline{K_v}(x) = \frac{K_v(x)}{x}$
minimum efficient scale x_{opt}	$\overline{K}'(x_{opt}) = 0$ (minimum of $\overline{K}$)
long-term break-even price (cost-covering price)	$\overline{K}(x_{opt})$
operating minimum x_{min}	$\overline{K_v}'(x_{min}) = 0$ (minimum of $\overline{K_v}$)
short-term break-even price	$\overline{K_v}(x_{min})$
point of inflexion of the cost function	$K''(x) = 0$
progressive costs	$K''(x) > 0$
degressive costs	$K''(x) < 0$

price p	
price function of demand (price-demand function) p_N	$p_N(x)$
price function of supply p_A	$p_A(x)$
market equilibrium	$p_A(x) = p_N(x)$
ceiling price	$p_N(0)$
saturation amount	$p_N(x) = 0$

revenue function E	$E(x) = p \cdot x$ or $E(x) = p_N(x) \cdot x$
marginal revenue function E'	$E'(x)$

profit function G	$G(x) = E(x) - K(x)$
marginal profit function G'	$G'(x)$
break-even point x_u upper profit limit x_o	$G(x_u) = G(x_o) = 0$ where $x_u \leq x_o$
profit range	$[x_u, x_o]$
Cournot's point C	$C = (x_C, p_N(x_C))$ where $G'(x_C) = 0$

24 Technical and Scientific Basics

ϱ ... density	t ... time
m ... mass	s ... distance
V ... volume	v ... velocity
F ... force	a ... acceleration
W ... work done	v_0 ... initial velocity
P ... power	

density	$\varrho = \frac{m}{V}$
force	$F = m \cdot a$
work done	$W = F \cdot s$
power	$P = \frac{W}{t}$

Motion

velocity for uniform linear motion	$v = \frac{s}{t}$
velocity for uniformly accelerated linear motion	$v = a \cdot t + v_0$
velocity in terms of the time t	$v(t) = s'(t)$
acceleration in terms of the time t	$a(t) = v'(t) = s''(t)$

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